FINITE ELEMENT ANALYSIS OF RESONANT CAVITIES AND WAVEGUIDES USING A VECTOR POTENTIAL FORMULATION

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ABSTRACT

A general, finite element formulation of resonant field behavior is presented. The formulation uses the magnetic vector potential $\vec{A}$ and time-integrated electric scalar potential ψ as solutions variables. Two types of physical modes are found. "Microwave" modes with nonzero $\vec{A}$ and ψ components represent ordinary, high-frequency resonant behavior in the presence of inhomogeneous, anisotropic dielectrics. Because the **time-integral** *of the scalar potential is used, "electrostatic" modes are found at zero frequency with only ψ components. Unphysical "spurious" modes are eliminated using a modal transformation method–in a way that does not affect physical solutions. The capability is demonstrated with two examples.*

INTRODUCTION

The numerical analysis of electromagnetic fields under resonant conditions is an important tool in the design of many microwave devices, including dielectric-loaded waveguides and cavities. Finite element and integral equation methods have been reported, which use both fields (1,2) and potential functions (3) as solutions variables. Often, TM and TE must be treated separately (3,4). In addition, most methods suffer from unphysical, "spurious" mode problems; though recent attempts to suppress these modes have achieved some success (4,2). We present a new finite element capability for analyzing the resonant properties of cavities and waveguides based on potential functions. This formulation obtains all modes with a single analysis, and does not suffer from spurious mode problems.

THE FORMULATION

The objective of resonant field analysis is to calculate the distribution of electric and magnetic fields, $\vec{E}$ and $\vec{B}$, under resonant conditions. The fields themselves are not convenient solution variables; because they are often discontinuous (for example, when sources are present). Thus potential functions are used to describe the magnetic field:

$$\vec{B} = \nabla \times \vec{A}. \tag{1}$$

To treat the general case, the time-integral of the electric scalar potential is added (5):

$$\vec{E} = -\nabla\dot{\psi} - \dot{\vec{A}}. \tag{2}$$

Use of the time-integrated scalar potential assures that the final matrix equation will be sparse, banded and symmetric. This formulation has been incorporated into a general purpose electromagnetic field analysis program, MSC/EMAS.

To obey Maxwell's equations, discrete values of $\vec{A}$ and ψ at predetermined grid points (*degrees-of-freedom*, or DOFs) must obey the following matrix equation (5):

$$\left[\begin{array}{c|c} M^{AA} & M^{A\psi} \\ \hline M^{\psi A} & M^{\psi\psi} \end{array}\right] \left\{\begin{array}{c} \ddot{\vec{A}} \\ \hline \ddot{\psi} \end{array}\right\} + \left[\begin{array}{c|c} B^{AA} & B^{A\psi} \\ \hline B^{\psi A} & B^{\psi\psi} \end{array}\right] \left\{\begin{array}{c} \dot{\vec{A}} \\ \hline \dot{\psi} \end{array}\right\}$$

$$+ \left[\begin{array}{c|c} K^{AA} & 0 \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} \vec{A} \\ \hline \psi \end{array}\right\} = \left\{\begin{array}{c} \vec{P}^A \\ \hline P^\psi \end{array}\right\}. \tag{3}$$

This matrix equation is equivalent to Maxwell's equations in their complete and general form. The $[M]$, $[B]$ and $[K]$ matrices represent dielectric, conductivity and permeability properties, respectively. Materials may be anisotropic; and in the case of magnetic materials, nonlinear. The column vector $\{u\}$ represents unknown $\vec{A}$ and ψ DOFs. The excitation vector $\{P\}$ is not used in resonant analysis.

An initial condition is also generated:

$$[M^{\psi\psi}]\{\dot{\psi}\} = \{\rho\} + \{\vec{D}_{normal}\} - [M^{\psi A}]\{\vec{A}^*\}, \tag{4}$$

where $\{\rho\}$ represents charge density; $\{\vec{D}_{normal}\}$ represents Neumann boundary conditions; and $\{\vec{A}^*\}$ represents an initial, divergence-free field distribution. This initial condition is just Gauss's law for electrostatics in matrix form.

Use of the vector potential brings with it some mathematical difficulties. It is well known that Maxwell's equations determine only the curl part of $\vec{A}$. Other parts, such as the divergence of $\vec{A}$, are undetermined. In numerical calculations these undetermined parts produce a **singular** $[K]$ matrix. For the particular case of eigenvalue analysis, singular $[K]$ produces a large number of *spurious modes*; modes with near-zero frequencies that cannot be easily distinguished from physical modes (6). Divergence singularities are removed by adding a *penalty energy* (7); so that $[K]$ is the sum of two parts:

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$$[K^{crl}] \propto (\nabla \times \vec{A}) \cdot [\nu](\nabla \times \vec{A}), \quad (5)$$

$$[K^{div}] \propto \alpha\nu(\nabla \cdot \vec{A})^2; \quad (6)$$

where the *penalty parameter* α is dimensionless. In a "weak" mathematical sense, the penalty energy requires the solution to satisfy the *Coulomb gauge*. In terms of modes, the penalty energy shifts divergence modes to higher frequencies.

Ideally, the divergence and curl parts of $\vec{A}$ are uncoupled; so that physical (curl) solutions are not affected by the choice of α. Due to practical element limitations, there is a weak coupling (less than one part in 10^7) between divergence and curl. This produces a small, unpredictable increase in the resonant frequencies of physical solutions with increasing α.

Use of the time-integrated electric scalar potential has an odd consequence; electrostatics is represented by a **dynamic** process. The initial potential $\dot{\psi}$ is determined by the initial condition, Eq. (4). Subsequent behavior is determined from solutions to the dynamic matrix equation, Eq. (3). Under electrostatic conditions ($\vec{A} = 0$, $[B] = 0$), the initial distribution of $\dot{\psi}$ remains unchanged; so the conventional electrostatic potential $\phi = \dot{\psi}$ is a time-independent function of position (as expected). This behavior has a significant effect on resonant analysis.

EIGENVALUE SOLUTIONS

With no conductivity ($[B] = 0$), no excitations ($\{P\} = 0$), and assuming $e^{i\omega t}$ time dependence, Eq. (3) reduces to an eigenvalue problem:

$$[\, -\omega_n^2 M \; + \; K \,]\{\xi_n\} = 0. \quad (7)$$

In bounded problems, solutions are obtained only at discrete *eigenvalues* ω_n^2. The associated *eigenvectors* $\{\xi_n\}$ are real. When inhomogeneous dielectrics are present, the eigenvectors must contain both $\vec{A}$ and ψ DOFs in order to account for the effects of electrical charge. Because potential functions are used, there is no distinction between TE, TM and hybrid modes; all mode types are obtained in a single calculation.

Electrostatic behavior is included in resonance problems when ψ DOFs are present. "Electrostatic" modes appear at exactly zero frequency; with all $\vec{A}$ DOFs equal to zero. These modes are orthogonal, linear combinations of electrostatic solutions of the matrix equation, Eq. (3); one mode for each unconstrained ψ DOF in the model. It must be emphasized: these are not "spurious" modes. They represent real, physical behavior. Since electrostatic behavior is not of interest in microwave devices, these modes are usually ignored.

The *mode spectrum* is shown schematically in Fig. 1. Many "electrostatic" modes are present at exactly zero frequency. Conventional high-frequency behavior is represented by "microwave" modes, in which both ψ and $\vec{A}$ are nonzero. The frequency of these modes is determined mainly by solutions to Maxwell's equations; but can be changed by very

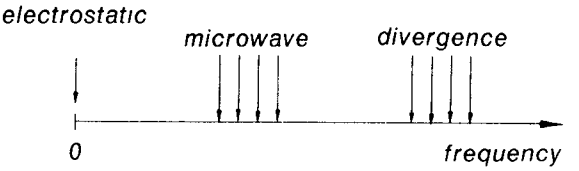

Figure 1: The mode spectrum in resonant analysis involves three types of modes: "electrostatic" modes, "microwave" modes and "divergence" modes.

large values of the penalty parameter. "Divergence" modes represent nonphysical behavior at higher frequencies associated with the divergence part of $\vec{A}$. The frequencies of these modes increase roughly as the square root of the penalty parameter. The grouping of modes into these three main types is thought to be general; and, apparently, occurs in all real eigenvalue analyses based on this formulation.

THE MODAL TRANSFORMATION METHOD

To be useful, the physical solutions of the eigenvalue problem, Eq. (7), must be: 1) easily distinguished from spurious solutions, and 2) must be independent of the penalty parameter. This is accomplished using a modal transformation method (8). First, "microwave" solutions to Eq. (7), $\{\xi_n\}$, are determined. These modes are not adequate solutions; because their frequencies depend slightly on α. However, they are ideal *basis functions* for expanding the actual solutions because they are divergence-free. The actual solutions $\{u\}$ are expressed as a sum of "microwave" modes;

$$\{u\} = \sum_n q_n\{\xi_n\}. \quad (8)$$

This expansion is substituted back into Eq. (7), and the resulting (very small) eigenvalue problem is solved for the *modal amplitudes* q_n. The eigenvalues are the same as those of the original problem, while the eigenvectors are recovered through Eq. (8). During this second solution, the penalty parameter is set to zero. As a result, the final solution is independent of α.

EXAMPLES

The first example, a dielectric-loaded waveguide, illustrates two-dimensional cutoff frequency calculations in inhomogeneous waveguides. The second, a spherical resonator, demonstrates mode calculations in three-dimensions. The first few modes of each model were obtained using a Lanczos eigenvalue algorithm. Perfectly conducting boundaries were simulated by constraining the tangent components of $\vec{A}$ and ψ to zero. In all cases, $\epsilon_o = \nu_o = 0$; and the divergence penalty parameter was 10^4. Results were later scaled to MKS values. Plotting and data recovery were performed using MSC/XL, a graphic pre- and postprocessor designed for electromagnetic applications.

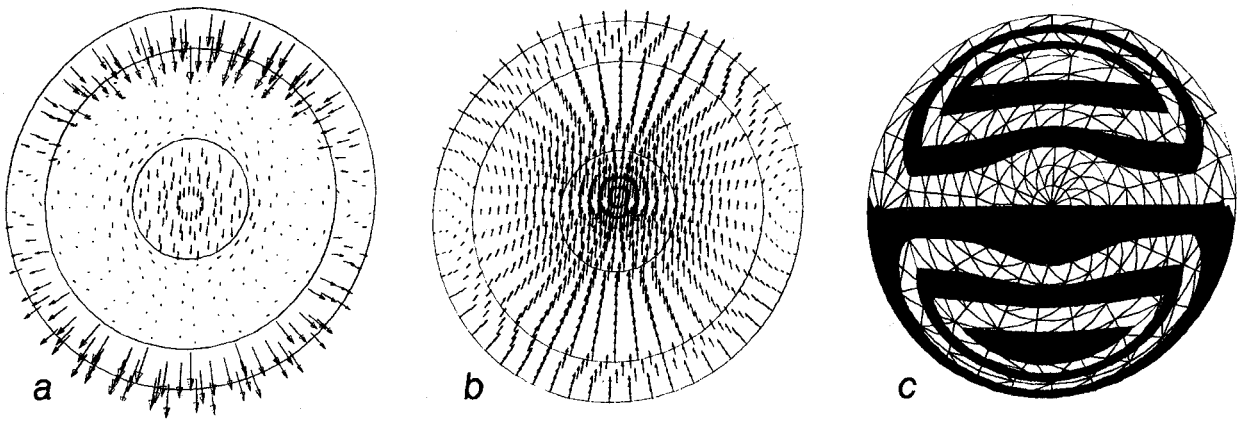

Figure 2: The electric field (a), vector potential (b) and scalar potential (c) in the HE_{12} mode of a dielectric-loaded waveguide.

Three-Layer Dielectric-Loaded Waveguide

A three-layer dielectric-loaded waveguide consists of a circular guide (radius=1.27 cm) containing a hollow dielectric rod (ϵ = 37.6) in the region 0.4 cm < radius < 1.0 cm. Approximately 450 6-node triangular elements were used to represent the two-dimensional cross section.

The cutoff frequencies are shown in Table 1 for the first ten modes. The error is also shown, relative to results published by Lin, *et. al.* (9). In all cases, errors are less than one per cent. Modes four and five are a degenerate pair of physical modes not listed by Lin *et. al.*

Results for the third mode, HE_{12}, are shown in Fig. 2. The arrow plot of the electric field shows that $\vec{E}$ lies within the plane; and is largest between the rod and the outer wall. The hollow center also contains substantial electric fields. For the same mode, arrow plots of $\vec{A}$ show little influence of the dielectric. Instead, the effects of dipole charges at dielectric boundaries are represented by the scalar potential, as can be seen in the contour plot of ψ. This is not surprising since, under the Coulomb gauge, $\vec{A}$ represents divergence-free solutions to the wave equation, while ψ represents electrostatic solutions.

Spherical Resonator

A 1 m diameter spherical resonator is modeled with 768 hexahedron and pentahedron elements connected to 795 grid points. The resulting eigenvalue problem involved 1901 unconstrained DOFs. The desired modes were extracted using a VAX 6000 computer (7 MIPS) in 742 CPU seconds.

Table 2 shows the resonant frequencies for the first eleven modes, error relative to theoretical results (10), and mode type. Higher-order modes show reduced accuracy; because the spatial dependence of the fields is more rapid relative to the fixed mesh density. We do not know the nomenclature for the eighth mode; but it resembles higher-order modes found in cylindrical resonators (10).

no	frequency (GHz)	relative error (%)	type
1	76.35937	-0.71	HE_{11}
2	93.60601	+0.26	HE_{12}
3	93.60793	+0.26	HE_{12}
4	131.5472		?
5	131.7642		?
6	156.4698	-0.37	HE_{13}
7	156.4715	-0.37	HE_{13}
8	157.7712	+0.42	HE_{13}
9	157.7132	+0.42	HE_{13}
10	163.7523	+0.36	HE_{14}

Table 1: Cutoff frequency, relative error and mode type for the first ten modes of a dielectric-loaded waveguide.

no	frequency (MHz)	relative error (%)	type
1	261.8659	+0.015	TM_{101}
2	265.1345	+1.26	TM_{101}
3	265.9924	+1.59	TM_{101}
4	367.4110	-0.56	TM_{201}
5	370.4039	+0.25	TM_{201}
6	373.8854	+1.19	TM_{201}
7	373.8854	+1.19	TM_{201}
8	375.0030	?	?
9	438.2904	+2.21	TE_{101}
10	438.9949	+2.38	TE_{101}
11	438.9949	+2.38	TE_{101}

Table 2: Frequency, error relative to theory, and mode type for the first eleven modes of a spherical resonator.

CONCLUSIONS

A comprehensive resonant analysis capability has been derived from a general formulation of finite element field analysis. By using vector and scalar potential functions as solution variables, the formulation can represent arbitrary geometries containing inhomogeneous, anisotropic dielectrics. All mode types are obtained from a single analysis without spurious mode difficulties. Two types of physical modes are found. "Electrostatic" modes represent electrostatic behavior at zero frequency, while "microwave" modes represent high-frequency resonant behavior. Spurious "divergence" modes are shifted to still higher frequencies (out of the range of interest) in a way that does not affect physical modes. The primary applications are in the computer-aided design of nonuniformly-loaded cavities and waveguides.

REFERENCES

1. A. Konrad, "A Direct Three-Dimensional Finite Element Method for the Solution of Electromagnetic Fields in Cavities," IEEE Trans **MAG-21**, 2276 (1985).

2. J. A. M. Svedin, "A Numerically Efficient Finite-Element Formulation for the General Waveguide Problem Without Spurious Modes," IEEE Trans **MTT-37**, 1708 (1989).

3. M. Swaminathan, E Arvas, T. K. Sarkar and A. R. Kjordjevic, "Computation of Cutoff Wavenumbers of TE and TM Modes in Waveguides of Arbitrary Cross Sections Using a Surface Integral Formulation," IEEE Trans **MTT-38**, 154 (1990).

4. T. Angkaew, M. Matsuhara, and N. Kumagai, "Finite element Analysis of Waveguide Modes: A Novel Approach That Eliminates Spurious Modes," IEEE Trans. **MTT-35**, 117 (1987).

5. B. E. MacNeal, J. R. Brauer, and R. N. Coppolino, "A General Finite Element Vector Potential Formulation of Electromagnetics using a Time-Integrated Electric Scalar Potential," IEEE Trans. **MAG-26**, 1768 (1990).

6. B. E. MacNeal, R. H. MacNeal and R. N. Coppolino, "Spurious Modes of Electromagnetic Vector Potential Finite Elements," IEEE Trans. **MAG-25**, 4141(1989).

7. J. P. Webb, "Efficient Generation of Divergence-Free Fields for the Finite Element Analysis of 3D Cavity Resonance," IEEE Trans **MAG-24**, 162 (1988).

8. B. E. MacNeal, L. A. Larkin, J. R. Brauer and A. O. Cifuentes, "Elimination of Finite Element Spurious Modes using a Modal Transformation Technique", IEEE Trans. **MAG-26**, 1765 (1990).

9. H. Lin and K. A. Zaki, "Properties of Three-Layer Dielectric Loaded Waveguides," IEEE Trans **MAG-25**, 2950 (1989).

10. J. A. Stratton, **Electromagnetic Theory**, McGraw-Hill Book Company, Inc., New York (1941).